Wavy fronts and speed bifurcation in excitable systems with cross diffusion

E. P. Zemskov,[1,*] K. Kassner,[2,†] and M. J. B. Hauser[3,‡]
[1]*Computing Centre of the Russian Academy of Sciences, Vavilova 40, 119333 Moscow, Russia*
[2]*Institut für Theoretische Physik, Otto-von-Guericke-Universität, Universitätsplatz 2, 39106 Magdeburg, Germany*
[3]*Institut für Experimentelle Physik, Otto-von-Guericke-Universität, Universitätsplatz 2, 39106 Magdeburg, Germany*
(Received 24 October 2007; published 25 March 2008)

A bifurcation of excitation fronts induced by cross diffusion in two-component bistable reaction-diffusion systems of activator-inhibitor type is discovered. This bifurcation is similar to the nonequilibrium Ising-Bloch bifurcation. A different type of fronts, whose spatial profiles are characterized by oscillating tails, are associated with this bifurcation. These fronts are described using exact analytical solutions of piecewise linear reaction-diffusion equations.

DOI: 10.1103/PhysRevE.77.036219 PACS number(s): 82.40.Bj, 05.45.−a, 82.40.Ck, 87.10.−e

I. REACTION-DIFFUSION SYSTEMS WITH SELF- AND CROSS DIFFUSION

Excitation waves in active media are usually described by a set of reaction-diffusion equations [1]. FitzHugh and Nagumo *et al.* [2] suggested a simple two-component system which is now used in the literature in its diffusive form

$$\frac{\partial u}{\partial t} = u(1-u)(u-a) - v + D_u \frac{\partial^2 u}{\partial x^2},$$

$$\frac{\partial v}{\partial t} = \varepsilon(u - \mu v) + D_v \frac{\partial^2 v}{\partial x^2}, \quad (1)$$

where $0<a<1$ and ε,μ are positive constants; D_u and D_v are the Fickian diffusion coefficients, hence non-negative constants. The constant a is an excitation threshold and ε is the ratio of time scales of the two reactions. This system is considered a basic model of excitation and propagation in various active physical, chemical, and biological media [3]. A further simplification consists of using the piecewise linear function $-u-v+\theta(u-a)$ [4] to model the nonlinear reaction terms. Using such an approximation for the FitzHugh-Nagumo model with a diffusion term in the first equation (i.e., with $D_v=0$), Rinzel and Keller [5] performed analytical calculations of wave propagation and found speed versus excitation threshold relations. Ito and Ohta [6] investigated the Rinzel-Keller model with double diffusive components. In all of these studies, traveling waves were always nonoscillatory in space. However, waves with oscillating tails may also occur in two-component reaction-diffusion models of this type [7].

In the present paper we continue our studies [7] of the dynamics of reaction-diffusion systems where self- and cross diffusion are explicitly taken into account. Here, we focus on solutions for a two-component system with self- and cross diffusion [8] that show fronts with oscillatory tails. In the context of population dynamics [8–10], cross-diffusion terms appear due to taxis processes, which are accepted as a mathematical description for the predator-prey pursuit and evasion systems [3]. The model proposed by Shigesada *et al.* [9] for spatial segregation of interacting species is given by the system of two equations

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}[(r_{10}+r_{11}u+r_{12}v)u] + (s_{10}-s_{11}u-s_{12}v)u,$$

$$\frac{\partial v}{\partial t} = \frac{\partial^2}{\partial x^2}[(r_{20}+r_{21}u+r_{22}v)v] + (s_{20}-s_{21}u-s_{22}v)v, \quad (2)$$

where all the r_{ij} and s_{ij} are constants. Models with cross diffusion also arise in the study of transport in magnetically confined fusion plasmas [11].

Spatial interactions in reaction-diffusion systems with cross diffusion are described by self-diffusion terms $\partial^2 u/\partial x^2$ and $\partial^2 v/\partial x^2$ and taxis terms $\partial/\partial x(u\partial v/\partial x)$ and $\partial/\partial x(v\partial u/\partial x)$ [8]. In an ideal multicomponent system, all components are similar to each other and all binary diffusion coefficients are constant. For simplicity, we consider here cross-diffusion terms having a form similar to the self-diffusion. This has also been done by Dubey *et al.* [10]. Then the simplest general description of the excitable medium by a two-component reaction-diffusion system includes both self- and cross diffusion in the following way:

$$\frac{\partial u}{\partial t} = f(u,v) + D_u \frac{\partial^2 u}{\partial x^2} + h_v \frac{\partial^2 v}{\partial x^2},$$

$$\frac{\partial v}{\partial t} = g(u,v) + D_v \frac{\partial^2 v}{\partial x^2} - h_u \frac{\partial^2 u}{\partial x^2}, \quad (3)$$

where $f(u,v)$ and $g(u,v)$ are the reaction terms. The activator reaction term $f(u,v)$ is usually a nonlinear function, whereas the inhibitor reaction $g(u,v)$ is often linear; $D_{u,v}$ represent the self-diffusion coefficients for the activator and inhibitor, respectively; for the cross-diffusion terms, the constants h_u and h_v are positive parameters associated with the retreat and pursuit of the prey and predator as a consequence of the interaction [3]. The choice of signs at $h_{u,v}$ mimicks the pursuit-evasion interaction of predator and prey [8].

*zemskov@ccas.ru, e-zemskov@yandex.ru
†klaus.kassner@ovgu.de
‡marcus.hauser@ovgu.de

To deal with solitary population waves we choose bistable dynamics. Thus we wish to consider the modified double diffusive Rinzel-Keller model [5], where the usual self-diffusion terms are supplemented with cross-diffusion ones. We aim at obtaining analytical solutions for fronts and determining their velocities.

II. CROSS DIFFUSION IN BOTH EQUATIONS: PROPAGATING FRONTS

Taking into account the form of the reaction functions for the bistable regime of the Rinzel-Keller system [5] with equal diffusion coefficients $D_u=D_v\equiv D$ and $h_u=h_v\equiv h$, the model equations read

$$\frac{\partial u}{\partial t}=-u-v+\theta(u-a)+D\frac{\partial^2 u}{\partial x^2}+h\frac{\partial^2 v}{\partial x^2},$$

$$\frac{\partial v}{\partial t}=\varepsilon(u-v)+D\frac{\partial^2 v}{\partial x^2}-h\frac{\partial^2 u}{\partial x^2}. \quad (4)$$

We wish to investigate traveling waves $u=u(\xi)$ and $v=v(\xi)$, where $\xi=x-ct$ is the traveling wave coordinate and c is the wave propagation speed. For the system (4), the fully analytical solutions are easily available in the particular case of $\varepsilon=1$.

In the following we give some details of the mathematical procedure for the derivation of the solutions. The general solutions $u(\xi)$ and $v(\xi)$ have the form

$$u(\xi)=\sum_n A_n e^{\lambda_n \xi}+u^*,$$

$$v(\xi)=\sum_n B_n e^{\lambda_n \xi}+v^*, \quad (5)$$

where A_n, B_n, u^*, and v^* are constants to be determined in each of the regions $u<a$ and $u>a$. The constants B_n can be expressed via the constants A_n. These expressions will be formulated below.

A. Front solutions and wave speed diagram

Inserting the general solutions (5) into the model equations (4) we obtain the following matrix equation:

$$\begin{pmatrix} D\lambda^2+c\lambda-1 & h\lambda^2-1 \\ -h\lambda^2+1 & D\lambda^2+c\lambda-1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}=0. \quad (6)$$

Hence the characteristic equation reads $(D\lambda^2+c\lambda-1)^2-i^2(h\lambda^2-1)^2=0$ (with $i^2=-1$) and yields four eigenvalues

$$\lambda_\pm^*=-p-iq\pm\sqrt{b+id}=-p-iq\pm y\pm iz,$$

$$\lambda_\pm=-p+iq\pm\sqrt{b-id}=-p+iq\pm y\mp iz, \quad (7)$$

where

$$p=\frac{cD}{2(D^2+h^2)},\quad q=\frac{ch}{2(D^2+h^2)},$$

$$b=p^2-q^2+\frac{p+q}{c/2},\quad d=2pq-\frac{p-q}{c/2},$$

$$y=\sqrt{(\sqrt{b^2+d^2}+b)/2},\quad z=\sqrt{(\sqrt{b^2+d^2}-b)/2}. \quad (8)$$

Therefore the front solutions read

$$u_1(\xi)=e^{k_+\xi}[A_1\cos(l_-\xi)+A_3\sin(l_-\xi)],$$

$$u_2(\xi)=e^{k_-\xi}[A_2\cos(l_+\xi)+A_4\sin(l_+\xi)]+1/2,$$

$$v_1(\xi)=e^{k_+\xi}[B_1\cos(l_-\xi)+B_3\sin(l_-\xi)],$$

$$v_2(\xi)=e^{k_-\xi}[B_2\cos(l_+\xi)+B_4\sin(l_+\xi)]+1/2, \quad (9)$$

where the notations $k_\pm$ and $l_\pm$ stand for $k_\pm=\pm y-p$ and $l_\pm=z\pm q$, respectively. The integration constants B are expressed as

$$B_{1,3}=-\frac{1}{\gamma_1^2+\delta_1^2}[(\alpha_1\gamma_1+\beta_1\delta_1)A_{1,3}\mp(\alpha_1\delta_1-\beta_1\gamma_1)A_{3,1}],$$

$$B_{2,4}=-\frac{1}{\gamma_2^2+\delta_2^2}[(\alpha_2\gamma_2+\beta_2\delta_2)A_{2,4}\mp(\alpha_2\delta_2-\beta_2\gamma_2)A_{4,2}] \quad (10)$$

with

$$\alpha_1=D(k_+^2-l_-^2)+ck_+-1,\quad \beta_1=l_-(2Dk_++c),$$

$$\gamma_1=h(k_+^2-l_-^2)-1,\quad \delta_1=2hk_+l_-,$$

$$\alpha_2=D(k_-^2-l_+^2)+ck_--1,\quad \beta_2=l_+(2Dk_-+c),$$

$$\gamma_2=h(k_-^2-l_+^2)-1,\quad \delta_2=2hk_-l_+. \quad (11)$$

There are five matching equations (two equations for functions u and v, two for their derivatives, and the fifth equation fixes the matching point) for five unknowns $(A_1,\ldots,A_4,c)$. From these equations, the speed relationship may be obtained by eliminating the $A_n, n=1,\ldots,4$.

The behavior of the speed c versus the excitation threshold a is illustrated for different values of the cross-diffusion constant h in Fig. 1. As the influence of the cross diffusion grows, a bifurcation in the behavior of the speed takes place. This bifurcation separates domains where the $c-a$ relation has a single value [Fig. 1(a)] from a domain where this relation is multivalued [Fig. 1(c)]. As this bifurcation is crossed, the $c-a$ curve folds to form three connected branches. The upper and the lower branches correspond to two counterpropagating fronts and terminate at certain critical values of a. A similar bifurcation scenario exists for the same system without cross diffusion, where the bifurcation parameter is the time scale ε. This bifurcation has been referred to in the literature as a nonequilibrium Ising-Bloch bifurcation [12,13].

Front profiles are presented graphically in Fig. 2. When the front has a nonzero speed, oscillations in the front profile are pronounced [Fig. 2(c)]. We denote these profiles as *wavy*

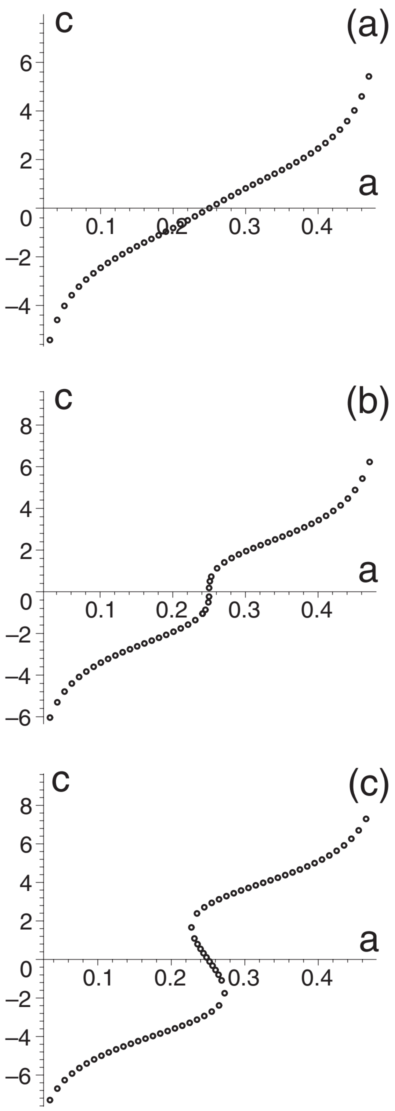

FIG. 1. Speed c versus excitation threshold a relation at different values of the cross-diffusion constant: (a) weak cross-diffusion case with $D=1$, $h=0.01$, (b) equal self- and cross-diffusion case with $D=h=1$, and (c) strong cross-diffusion case with $D=1$, $h=3$.

fronts. When the front is motionless, the tails of the front show nonoscillatory behavior. For moving fronts, we present examples of front solutions with positive values of the speed. Therefore the fronts move from left to right, i.e., when the cross-diffusion front propagates, the pronounced spatial oscillations precede the front. In the model where only self-diffusion is present the pronounced oscillations (if any) lag behind the front. The generic form of the oscillatory behavior (spiral in the $u-v$ diagram [Fig. 2(f)]) remains similar in both cases.

B. Linear stability analysis

To investigate the stability of the fronts $u(\xi)$ and $v(\xi)$ we consider perturbed solutions of the form $\Delta u(\xi,\chi,t)=u(\xi)+\tilde{u}(\xi)\exp(\omega t+i\kappa\chi)$ and $\Delta v(\xi,\chi,t)=v(\xi)+\tilde{v}(\xi)\exp(\omega t+i\kappa\chi)$, where χ is the direction transverse to the direction of propagation of the planar front. A linear stability analysis assumes perturbed solutions of the form

$$U(\xi,\chi,t)=u(\xi)+\Delta u(\xi,\chi,t),$$

$$V(\xi,\chi,t)=v(\xi)+\Delta v(\xi,\chi,t), \tag{12}$$

where small perturbations $\Delta u(\xi,\chi,t)$ and $\Delta v(\xi,\chi,t)$ are added to the planar front solutions. In the stationary frame, the full perturbed solutions satisfy

$$\frac{\partial U}{\partial t}=D\left(\frac{\partial^2 U}{\partial \xi^2}+\frac{\partial^2 U}{\partial \chi^2}\right)+h\left(\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \chi^2}\right)+f(U,V),$$

$$\frac{\partial V}{\partial t}=D\left(\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \chi^2}\right)-h\left(\frac{\partial^2 U}{\partial \xi^2}+\frac{\partial^2 U}{\partial \chi^2}\right)+g(U,V) \tag{13}$$

and $u(\xi)$, $v(\xi)$ are time-independent solutions of these equations. Subtracting the equations for the unperturbed solutions and linearizing for small Δu and Δv, we obtain the variational equations for the perturbations

$$\frac{\partial \Delta u}{\partial t}=D\left(\frac{\partial^2 \Delta u}{\partial \xi^2}+\frac{\partial^2 \Delta u}{\partial \chi^2}\right)+h\left(\frac{\partial^2 \Delta v}{\partial \xi^2}+\frac{\partial^2 \Delta v}{\partial \chi^2}\right)+\frac{\partial f}{\partial u}\Delta u + \frac{\partial f}{\partial v}\Delta v,$$

$$\frac{\partial \Delta v}{\partial t}=D\left(\frac{\partial^2 \Delta v}{\partial \xi^2}+\frac{\partial^2 \Delta v}{\partial \chi^2}\right)-h\left(\frac{\partial^2 \Delta u}{\partial \xi^2}+\frac{\partial^2 \Delta u}{\partial \chi^2}\right)+\frac{\partial g}{\partial u}\Delta u + \frac{\partial g}{\partial v}\Delta v. \tag{14}$$

The equations for the eigenfunctions (the variational equations) read

$$D\frac{d^2\tilde{u}}{d\xi^2}+h\frac{d^2\tilde{v}}{d\xi^2}-[\Omega+D\kappa^2-\delta(u-a)]\tilde{u}-(1+h\kappa^2)\tilde{v}=0,$$

$$D\frac{d^2\tilde{v}}{d\xi^2}-h\frac{d^2\tilde{u}}{d\xi^2}+(1+h\kappa^2)\tilde{u}-(\Omega+D\kappa^2)\tilde{v}=0, \tag{15}$$

where $\Omega=1+\omega$.

Inserting the perturbation solutions in the form $\tilde{u},\tilde{v}(\xi)=\Sigma\tilde{A},\tilde{B}\exp(\tilde{\lambda}\xi)$ into the variational equations (15) we obtain the following matrix equation:

$$\begin{pmatrix} D(\tilde{\lambda}^2-\kappa^2)-\Omega & h(\tilde{\lambda}^2-\kappa^2)-1 \\ -h(\tilde{\lambda}^2-\kappa^2)+1 & D(\tilde{\lambda}^2-\kappa^2)-\Omega \end{pmatrix}\begin{pmatrix}\tilde{A}\\ \tilde{B}\end{pmatrix}=0. \tag{16}$$

Hence the characteristic equation is $[D(\tilde{\lambda}^2-\kappa^2)-\Omega]^2-i^2[h(\tilde{\lambda}^2-\kappa^2)-1]^2=0$ and then

$$\tilde{\lambda}^2=\frac{D\Omega+h}{D^2+h^2}+\kappa^2\pm i\frac{D-h\Omega}{D^2+h^2}. \tag{17}$$

When the pure self-diffusion case ($D=1$ and $h=0$) is considered, then $\Omega=\tilde{\lambda}^2-\kappa^2\mp i$ and it can be seen that the fastest growing mode will always correspond to $\kappa=0$. We may then restrict ourselves to the $\kappa=0$ case as we did in Ref. [7].

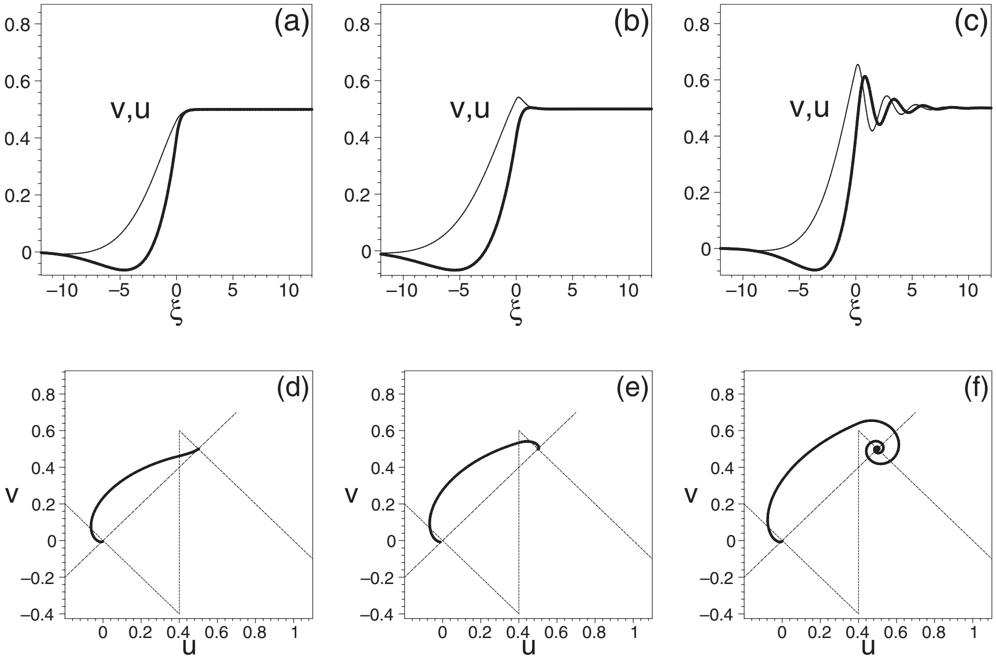

FIG. 2. Front profiles for [(a)–(c)] activator $u(\xi)$ (bold lines) and inhibitor $v(\xi)$ (thin lines) and [(d)–(f)] $u-v$ phase diagrams for [(a) and (d)] the weak cross-diffusion case with $D=1$, $h=0.01$, [(b) and (e)] equal self- and cross-diffusion case with $D=h=1$, and [(c) and (f)] strong cross-diffusion case with $D=0.01$, $h=1$. The value of the excitation threshold is fixed at $a=0.4$ so that the front speed is always positive. The null-clines $f(u,v)=0$ and $g(u,v)=0$ are shown by thin lines [(d)–(f)].

When the pure cross-diffusion case ($D=0$ and $h=1$) is considered, then $\Omega=\pm i(\tilde{\lambda}^2-\kappa^2-1)$ and it is evident that the real part of Ω is not changed by κ, which therefore does not affect the stability of the solutions. Thus we can turn our attention to the $\kappa=0$ case again.

Now we will consider the situation when the system is placed in the frame moving at velocity c. Then with the perturbations $\Delta u(\xi,t)=\tilde{u}(\xi)e^{\omega t}$, $\Delta v(\xi,t)=\tilde{v}(\xi)e^{\omega t}$ and the expressions for the null-clines, we write the variational equations as

$$D\frac{d^2\tilde{u}}{d\xi^2}+h\frac{d^2\tilde{v}}{d\xi^2}+c\frac{d\tilde{u}}{d\xi}-[1+\omega-\delta(u-a)]\tilde{u}-\tilde{v}=0,$$

$$D\frac{d^2\tilde{v}}{d\xi^2}-h\frac{d^2\tilde{u}}{d\xi^2}+c\frac{d\tilde{v}}{d\xi}+\tilde{u}-(1+\omega)\tilde{v}=0. \qquad (18)$$

Inserting the perturbation solutions into the variational equations (18) we obtain the following matrix equation:

$$\begin{pmatrix} D\tilde{\lambda}^2+c\tilde{\lambda}-\Omega & h\tilde{\lambda}^2-1 \\ -h\tilde{\lambda}^2+1 & D\tilde{\lambda}^2+c\tilde{\lambda}-\Omega \end{pmatrix}\begin{pmatrix}\tilde{A}\\ \tilde{B}\end{pmatrix}=0. \qquad (19)$$

Then the characteristic equation written as $(D\tilde{\lambda}^2+c\tilde{\lambda}-\Omega)^2-i^2(h\tilde{\lambda}^2-1)^2=0$ yields four eigenvalues,

$$\tilde{\lambda}^*_\pm=-\tilde{p}-i\tilde{q}\pm\tilde{y}\pm i\tilde{z},$$

$$\tilde{\lambda}_\pm=-\tilde{p}+i\tilde{q}\pm\tilde{y}\mp i\tilde{z}, \qquad (20)$$

where

$$\tilde{p}=p, \quad \tilde{q}=q,$$

$$\tilde{b}=\tilde{p}^2-\tilde{q}^2+\frac{\tilde{p}\Omega+\tilde{q}}{c/2}, \quad \tilde{d}=2\tilde{p}\tilde{q}-\frac{\tilde{p}-\tilde{q}\Omega}{c/2},$$

$$\tilde{y}=\sqrt{(\sqrt{\tilde{b}^2+\tilde{d}^2}+\tilde{b})/2}, \quad \tilde{z}=\sqrt{(\sqrt{\tilde{b}^2+\tilde{d}^2}-\tilde{b})/2}. \qquad (21)$$

The perturbation solutions read

$$\tilde{u}_1(\xi)=e^{\tilde{k}_+\xi}[\tilde{A}_1\cos(\tilde{l}_-\xi)+\tilde{A}_3\sin(\tilde{l}_-\xi)],$$

$$\tilde{u}_2(\xi)=e^{\tilde{k}_-\xi}[\tilde{A}_2\cos(\tilde{l}_+\xi)+\tilde{A}_4\sin(\tilde{l}_+\xi)],$$

$$\tilde{v}_1(\xi)=e^{\tilde{k}_+\xi}[\tilde{B}_1\cos(\tilde{l}_-\xi)+\tilde{B}_3\sin(\tilde{l}_-\xi)],$$

$$\tilde{v}_2(\xi)=e^{\tilde{k}_-\xi}[\tilde{B}_2\cos(\tilde{l}_+\xi)+\tilde{B}_4\sin(\tilde{l}_+\xi)] \qquad (22)$$

using the notations $\tilde{k}_\pm=\pm\tilde{y}-\tilde{p}$ and $\tilde{l}_\pm=\tilde{z}\pm\tilde{q}$. The integration constants $\tilde{B}$ are

$$\tilde{B}_{1,3}=-\frac{1}{\tilde{\gamma}_1^2+\tilde{\delta}_1^2}[(\tilde{\alpha}_1\tilde{\gamma}_1+\tilde{\beta}_1\tilde{\delta}_1)\tilde{A}_{1,3}\mp(\tilde{\alpha}_1\tilde{\delta}_1-\tilde{\beta}_1\tilde{\gamma}_1)\tilde{A}_{3,1}],$$

$$\tilde{B}_{2,4}=-\frac{1}{\tilde{\gamma}_2^2+\tilde{\delta}_2^2}[(\tilde{\alpha}_2\tilde{\gamma}_2+\tilde{\beta}_2\tilde{\delta}_2)\tilde{A}_{2,4}\mp(\tilde{\alpha}_2\tilde{\delta}_2-\tilde{\beta}_2\tilde{\gamma}_2)\tilde{A}_{4,2}] \qquad (23)$$

with

$$\tilde{\alpha}_1=D(\tilde{k}_+^2-\tilde{l}_-^2)+c\tilde{k}_+-\Omega, \quad \tilde{\beta}_1=\tilde{l}_-(2D\tilde{k}_++c),$$

$$\tilde{\gamma}_1 = h(\tilde{k}_+^2 - \tilde{l}_-^2) - 1, \quad \tilde{\delta}_1 = 2h\tilde{k}_+\tilde{l}_-,$$

$$\tilde{\alpha}_2 = D(\tilde{k}_-^2 - \tilde{l}_+^2) + c\tilde{k}_- - \Omega, \quad \tilde{\beta}_2 = \tilde{l}_+(2D\tilde{k}_- + c),$$

$$\tilde{\gamma}_2 = h(\tilde{k}_-^2 - \tilde{l}_+^2) - 1, \quad \tilde{\delta}_2 = 2h\tilde{k}_-\tilde{l}_+. \tag{24}$$

The matching conditions for the derivatives have jumps due to the delta function in the variational equations. To make use of this behavior we integrate the variational equations over a small interval about the matching point of the perturbations. Nonzero contributions arise from the diffusive terms and terms with a delta function, i.e.,

$$D\frac{d\tilde{u}}{d\xi}\Big|_{0-\epsilon}^{0+\epsilon} + h\frac{d\tilde{v}}{d\xi}\Big|_{0-\epsilon}^{0+\epsilon} + \int_{0-\epsilon}^{0+\epsilon}\frac{\delta(\xi)}{|du(0)/d\xi|}\tilde{u}d\xi = 0 \tag{25}$$

for the first equation in Eq. (18) and

$$D\frac{d\tilde{v}}{d\xi}\Big|_{0-\epsilon}^{0+\epsilon} - h\frac{d\tilde{u}}{d\xi}\Big|_{0-\epsilon}^{0+\epsilon} = 0 \tag{26}$$

for the second. After the integration of the delta function we can rewrite Eq. (25) as

$$D\left(\frac{d\tilde{u}_2}{d\xi} - \frac{d\tilde{u}_1}{d\xi}\right) + h\left(\frac{d\tilde{v}_2}{d\xi} - \frac{d\tilde{v}_1}{d\xi}\right) + \frac{\tilde{u}_0}{|du(0)/d\xi|} = 0, \tag{27}$$

where $\tilde{u}_0$=const is a perturbation amplitude, so that Eq. (26) reads

$$D\left(\frac{d\tilde{v}_2}{d\xi} - \frac{d\tilde{v}_1}{d\xi}\right) = h\left(\frac{d\tilde{u}_2}{d\xi} - \frac{d\tilde{u}_1}{d\xi}\right). \tag{28}$$

Thus in the absence of cross diffusion ($D=1, h=0$), there is a jump in the activator derivative

$$d\tilde{u}_1(0)/d\xi = d\tilde{u}_2(0)/d\xi + \tilde{u}_0/|du(0)/d\xi|,$$

$$d\tilde{v}_1(0)/d\xi = d\tilde{v}_2(0)/d\xi; \tag{29}$$

however, when the self-diffusion vanishes ($D=0, h=1$) the jump appears in the inhibitor derivative

$$d\tilde{u}_1(0)/d\xi = d\tilde{u}_2(0)/d\xi,$$

$$d\tilde{v}_1(0)/d\xi = d\tilde{v}_2(0)/d\xi + \tilde{u}_0/|du(0)/d\xi|. \tag{30}$$

The stability analysis shows that the middle branch of the multivalued curve in Fig. 1(c) is unstable (for $\kappa=0$), whereas the upper and lower branches are stable.

III. CROSS DIFFUSION IN ONE EQUATION: STATIONARY FRONTS

In the model described above, cross-diffusion terms appear in both equations of system (4). Now we will consider the situation when the cross diffusion needs to be taken into account only in a single equation. Such a description would be useful whenever the cross-diffusion coefficients are very different for the two components so that neglecting one is justified. Here, we assume the case where cross diffusion appears only in the activator equation. Hence our simplified model reads

$$\frac{\partial u}{\partial t} = -u - v + \theta(u-a) + D\frac{\partial^2 u}{\partial x^2} + h\frac{\partial^2 v}{\partial x^2},$$

$$\frac{\partial v}{\partial t} = u - v + D\frac{\partial^2 v}{\partial x^2}. \tag{31}$$

For the sake of analytical solvability, we restrict our consideration to stationary solutions. Then the matrix equation (6) takes the form

$$\begin{pmatrix} D\lambda^2 - 1 & h\lambda^2 - 1 \\ 1 & D\lambda^2 - 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0, \tag{32}$$

so that the characteristic equation yields the eigenvalues

$$\lambda_{1,\ldots,4} = \pm\frac{1}{\sqrt{2D}}\sqrt{2D + h \pm \sqrt{(2D+h)^2 - 8D^2}}. \tag{33}$$

Then it follows that, when $h/(2D) > \sqrt{2}-1$, the roots (33) are real, whereas when $0 < h/(2D) < \sqrt{2}-1$ the roots contain imaginary parts (taking into account that the constants D and h are positive).

When the λ_n are real, the front solutions are

$$u_1(x) = A_1 e^{\lambda_1 x} + A_3 e^{\lambda_3 x},$$

$$u_2(x) = A_2 e^{\lambda_2 x} + A_4 e^{\lambda_4 x} + 1/2,$$

$$v_1(x) = B_1 e^{\lambda_1 x} + B_3 e^{\lambda_3 x},$$

$$v_2(x) = B_2 e^{\lambda_2 x} + B_4 e^{\lambda_4 x} + 1/2 \tag{34}$$

and $B_n = A_n/(1 - D\lambda_n^2)$, $n=1,\ldots,4$. When the λ_n have imaginary parts, i.e.,

$$\lambda_{1,2} = \pm\sqrt{\bar{b} + i\bar{d}} = \pm\bar{y} \pm i\bar{z},$$

$$\lambda_{3,4} = \pm\sqrt{\bar{b} - i\bar{d}} = \pm\bar{y} \mp i\bar{z},$$

$$\bar{b} = \frac{2D+h}{2D^2}, \quad \bar{d} = \frac{1}{2D^2}\sqrt{8D^2 - (2D+h)^2},$$

$$\bar{y} = \sqrt{(\sqrt{\bar{b}^2 + \bar{d}^2} + \bar{b})/2}, \quad \bar{z} = \sqrt{(\sqrt{\bar{b}^2 + \bar{d}^2} - \bar{b})/2}, \tag{35}$$

the front solutions read

$$u_1(x) = e^{yx}[A_1\cos(zx) + A_3\sin(zx)],$$

$$u_2(x) = e^{-yx}[A_4\cos(zx) + A_2\sin(zx)] + 1/2,$$

$$v_1(x) = e^{yx}[B_1\cos(zx) + B_3\sin(zx)],$$

$$v_2(x) = e^{-yx}[B_4\cos(zx) + B_2\sin(zx)] + 1/2 \tag{36}$$

and the B_n constants are

$$B_{1,3} = -\frac{1}{\alpha^2 + \beta^2}(\alpha A_{1,3} \mp \beta A_{3,1}),$$

$$B_{2,4} = -\frac{1}{\alpha^2+\beta^2}(\alpha A_{2,4} \mp \beta A_{4,2}), \tag{37}$$

where $\alpha = Db-1 = h/(2D)$ and $\beta = Dd = \sqrt{2-[1+h/(2D)]^2}$. The stationary front profiles exhibit nonoscillating behavior (data not presented here). Another analytically solvable case of cross diffusion in one equation with $D_u = D$ and $D_v = h_v$ ($h_u = 0$) [see Eq. (3)] produces a characteristic equation of the form $(h\lambda^2-1)(D\lambda^2-2)=0$ and is beyond the scope of our investigation.

IV. REALISTIC MODELS AND THEIR APPLICATION

Cross-diffusion terms must be considered explicitly in certain reaction-diffusion systems occurring in nature. A typical example is combustion, where cross diffusion acts on all variables and contributes substantially to the dynamics of flame fronts [14,15]. For instance, in hydrogen-air flames at atmospheric pressure, the Fickian and cross-diffusion effects are comparable in magnitude [16]. Compared to experimental data, numerical simulations reveal that a neglect of cross-diffusion terms leads to an inaccurate estimation of the flame front thickness [15]. Population dynamics is another area where mutual cross diffusion plays a central role, especially in predator-prey systems, where the predator actively pursues the prey, while the prey tries to evade the predator [3].

Reaction-diffusion systems, where only one of the variables is affected by cross diffusion, are also known. An example is the $O_2 + H_2$ reaction to water on a Rh (110) surface doped with the promotor K [17,18]. The latter is a mobile species adsorbed on the Rh surface, and it enhances the rate of the catalytic reaction. In this bistable system, a reaction front propagates on the Rh support, which in addition to water formation causes a redistribution of the adsorbed potassium, since the latter is dragged along with the reaction fronts, leading to its accumulation. When two reducing fronts (i.e., reaction fronts invading oxygen-rich domains) collide, they annihilate and form islands of coadsorbed oxygen and potassium of macroscopic size. When the supply of gaseous reactants is switched off, diffusion processes reestablish a homogeneous distribution of K on the Rh surface [17,18]. The observed phenomena have been reproduced numerically to good agreement using a three-variable model, where cross diffusion has only been considered in the evolution of the potassium concentration [17,19].

To summarize, we have described a type of excitation waves which is characteristic for two-component reaction-diffusion systems. These waves show oscillations in the spatial profile. The characteristic feature of the considered oscillating tails is their damping behavior that distinguishes them from the classical periodic waves, which are fundamental solutions of oscillatory reaction-diffusion equations. The waves with damping oscillatory tails are related to an intermediate dynamical regime. This is similar to the oscillatory one, but appears in excitable systems, which are only "quasioscillatory." The oscillating waves considered here arise in the system with both self- and cross-diffusion terms in the equations. When only cross diffusion needs to be considered in one equation the oscillations in the wave profile vanish and the wave profiles become monotonic [20].

ACKNOWLEDGMENTS

E.P.Z. thanks M.A. Tsyganov for useful discussions and for introduction into excitable systems with cross diffusion and the German Academic Exchange Service (DAAD) for financial support (Wiedereinladungsstipendium 2007).

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